Characterizations of the Trace

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ABSTRACT

The paper contains a number of equivalent conditions which characterize the trace among the linear functionals on the matrix algebra. Some of these results are extended to more general operator algebras.

1. INTRODUCTION

Given a matrix A in $M_n(\mathbb{C})$, we list its eigenvalues so that $|\lambda_1(A)| \geqslant |\lambda_2(A)| \geqslant \cdots \geqslant |\lambda_n(A)|$. The singular values of A are then defined by $s_j(A) = \lambda_j(|A|)$, $j = 1,2,\dots,n$, where $|A| = (A^*A)^{1/2}$. For $k = 1,2,\dots$ we have the (special case of the) Schur-Weyl inequalities

$$|\lambda_1(A)|^k + \cdots + |\lambda_n(A)|^k \leqslant s_1(A)^k + \cdots + s_n(A)^k. \tag{1}$$

See, for instance, [9, p. 107], [12, p. 143], [13, p. 35], [19, p. 155], [20, p. 318] for various proofs; the books [19] and [20] also describe the history of these

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inequalities. It follows from (1) that

$$|\mathrm{tr}(A^k)| \leqslant \mathrm{tr}(|A|^k). \tag{2}$$

The principal motivation of this paper is to show that the Schur-Weyl inequalities are globally so sharp that their consequence (2), for any individual k, actually characterizes the trace among the linear functionals on the matrix algebra $M_n(\mathbb{C})$; see Theorem 1 for the precise formulation. Our proof extends the method of Gardner [8], who studied the case $k = 1$. The simplified proof in [27] does not apply to the case $k > 1$.

Other proofs of (2) can be derived in more general situations where the classical Schur-Weyl inequalities are not available. For instance, (2) appears as a consequence of the Hölder inequality in the theory of noncommutative integration [4, 5, 15]. We shall show that (2) is still equivalent to the trace property even in this general context (Theorem 5). In fact, it is the matrix case which is essential.

In Theorem 2 we list a number of other equivalent conditions characterizing the matrix trace, notably in terms of the behavior on the idempotent or nilpotent elements. Again some of these characterizations admit conceptual proofs in a general setting (Theorems 3 and 4).

2. MATRIX ALGEBRAS

THEOREM 1. *A linear functional f on $M_n(\mathbb{C})$ is a nonnegative scalar multiple of the trace if and only if the inequality*

$$|f(A^k)| \leqslant f(|A|^k) \tag{3}$$

holds for all matrices A and some fixed $k = 1, 2, \ldots$.

Proof. Suppose that f satisfies (3). Hence f is a positive functional. Since the hypothesis and the claim are invariant with respect to a unitary transformation, we may assume f in the standard representation $f(X) = \mathrm{tr}(SX)$, where the matrix S is diagonal with nonnegative entries; cf. [8], [9, p. 129]. We shall show that any two consecutive diagonal entries of S are equal. To this end we exploit the condition (3) for certain rank-one matrices in the corresponding subalgebra isomorphic to $M_2(\mathbb{C})$. So without loss of generality we assume that $n = 2$ and that the diagonal entries of S satisfy

$0 \leqslant s_{11} = t \leqslant s_{22} = 1$. As in [8], we apply (3) to the matrices

$$A = \begin{pmatrix} p & 1 \\ p^2 & p \end{pmatrix}, \qquad \text{so that} \quad |A| = \begin{pmatrix} p^2 & p \\ p & 1 \end{pmatrix},$$

where p is a real parameter. It is easy to check by induction that

$$A^k = 2^{k-1}p^{k-1}A, \qquad |A|^k = (p^2+1)^{k-1}|A|.$$

Hence we have

$$f(A^k) = 2^{k-1}p^k(t+1),$$

$$f(|A|^k) = (p^2+1)^{k-1}(tp^2+1),$$

so that the condition (3) yields

$$2^{k-1}p^k(t+1) \leqslant (p^2+1)^{k-1}(tp^2+1),$$

or, in other words,

$$tu(p) \leqslant v(p), \tag{4}$$

where

$$u(p) = 2^{k-1}p^k - (p^2+1)^{k-1}p^2,$$

$$v(p) = (p^2+1)^{k-1} - 2^{k-1}p^k.$$

We observe that $u(1) = v(1) = 0$; hence (4) for $p > 1$ implies

$$t\frac{u(p)-u(1)}{p-1} \leqslant \frac{v(p)-v(1)}{p-1}.$$

Letting $p \to 1_+$, we conclude that

$$tu'(1) \leqslant v'(1).$$

A direct calculation of the derivatives shows that $u'(1) = v'(1) = -2^{k-1}$, and the desired inequality $t \geqslant 1$ follows. ∎

In the next theorem we collect further conditions equivalent to the trace property. Most of them are well known or scattered in the literature. We believe it is useful to list them for the sake of completeness. The spectral radius of A is denoted by $r(A)$.

THEOREM 2. *Let f be a nonzero linear functional on $M_n(\mathbb{C})$. The following are equivalent*:

(a) *f is a constant multiple of the trace*;
(b) $f(AB - BA) = 0$ *for all A and B*;
(c) $f(CAC^{-1}) = f(A)$ *for all A and invertible C*;
(d) $|f(A)| \leqslant kr(A)$ *for some constant c and for all A*;
(e) $\limsup_{k\to\infty} |f(A^k)|^{1/k} = r(A)$ *for all A*;
(f) $f(P) \neq 0$ *for every nonzero idempotent P*;
(g) $\sup\{|f(P)|: P^2 = P\}$ *is finite*;
(h) *f is constant on idempotents of the same rank*;
(i) $f(A) = 0$ *whenever $A^2 = 0$*;
(j) $f(A) = f(A^2) = \cdots = f(A^n) = 0$ *implies A is nilpotent*.

Proof. It is enough to add a few bibliographic comments. That (b) characterizes the trace is a consequence of the Shoda theorem [22], which says that the commutators form a subspace of codimension one in $M_n(\mathbb{C})$, namely the kernel of the trace. However, (b) → (c) → (h) → (a) is a chain of easier implications. Property (e) of the trace was noticed by Wimmer [25]; the matrix

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

satisfies $A^3 = A$, which shows that "lim sup" in (e) cannot be replaced by "lim". The converse is a consequence of the implication (f) → (a), which was noticed by Johnson [10, p. 338]; see also Feroe [6]. We offer a simple conceptual proof in the next section. A proof of the implication (a) → (j) can be found in [7, p. 11]. The converse follows immediately via condition (f); see also [6]. For the implications (g) → (a) and (i) → (a) we were not able to find any reference in the literature; however, a simple direct proof may proceed as in Theorem 1 by using, in the latter case, nilpotent matrices having one principal submatrix like

$$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

and zeros otherwise. A similar argument using the idempotents

$$\begin{pmatrix} p & 1-p \\ p & 1-p \end{pmatrix}$$

and their transposes proves the implications (f) $\rightarrow$ (a) and (g) $\rightarrow$ (a). Finally, the implication (d) $\rightarrow$ (a) is an immediate consequence of the implications (g) $\rightarrow$ (a) or (i) $\rightarrow$ (a); a standard analytic proof of the implication (d) $\rightarrow$ (b) can be found in [3, p. 114]. ∎

To motivate the more general considerations in the next section we note that (h) is equivalent to

(h′) *f is constant on the connected components of the set of idempotent matrices;*

see [26, Theorem 3.3].

A characterization of the trace property in terms of the behavior on the orthogonal projections will be given in Theorem 4.

3. BANACH ALGEBRAS

PROPOSITION 1. *Let f be a linear functional on a Banach algebra. Suppose that $f(p) \neq 0$ for every nonzero idempotent p. Then f is constant on the connected components of the set of idempotents in the algebra. In other words, condition* (f) *implies* (h′) *in general. Also,* (g) *implies* (h′).

Proof. Let u and v be different idempotents with $\|u - v\| < 1$. Let $w = u(u + v - 1)^{-2}v$ be their Kovarik idempotent [14, p. 347]. Then the two lines, one passing through u and w, and the other through v and w, consist entirely of idempotents; see [14, p. 347]. Consequently, if the restriction of our linear functional to these lines were not constant, it would be unbounded and there would exist an idempotent p such that $f(p) = 0$. Of course, $p \neq 0$ because the two lines lie in the same connected component while 0 is an isolated idempotent. This shows that $f(u) = f(w) = f(v)$.

Since every pair of idempotents in the same component can be connected by a chain of idempotents where any two consecutive members have distance less than 1 (see [26, p. 179]), the proposition follows. ■

PROPOSITION 2. *Let f be a bounded linear functional on a Banach algebra satisfying condition* (h′). *Let a be an element in the closed linear span of the idempotents. Then $f(ab - ba) = 0$ for all b in the algebra.*

Proof. Given an idempotent p, we consider the entire function

$$f(e^{-\lambda b} p\, e^{\lambda b}) = f(p + \lambda(pb - bp) + \cdots).$$

Since the left-hand side is constant, we conclude that $f(pb - bp) = 0$. Since f is linear and bounded, this extends to the closed linear span of the idempotents; hence $f(ab - ba) = 0$. ■

THEOREM 3. *Let f be a bounded linear functional on a Banach algebra which is the closed linear span of its idempotents. Then the conditions* (b), (c), *and* (h′) *of Theorem* 2 *are equivalent.*

Proof. Clearly (b) implies (c). Next, (c) implies (h′) by [26, Theorem 3.3]. Finally, (h′) implies (b) by Proposition 2. ■

4. OPERATOR ALGEBRAS

Let $\mathscr{A}$ be a C^*-algebra. A positive linear functional f on $\mathscr{A}$ is said to be *tracial* if it satisfies property (b) of Theorem 2. In the characterizations of the tracial functionals the following lemma is of basic importance. It is implicitly contained in [8]. For its proof we recall that two projections (i.e. self-adjoint idempotents) p and q in a von Neumann algebra $\mathscr{M}$ are said to be equivalent when $p = v^*v$ and $q = vv^*$ for some element v in $\mathscr{M}$; cf. [11, p. 402].

LEMMA. *Let f be a positive linear functional on a von Neumann algebra $\mathscr{M}$. If f restricted to every subalgebra isomorphic to $M_2(\mathbf{C})$ is tracial, then f is tracial on $\mathscr{M}$.*

Proof. By [11, p. 505] it is enough to show that $f(p) = f(q)$ for any equivalent projections p and q in $\mathcal{M}$. If these are orthogonal ($pq = qp = 0$), then the subalgebra generated by the partial isometries v and v^*, accomplishing the equivalence between p and q, is isomorphic to $M_2(\mathbb{C})$, so that we have $f(p) = f(q)$ by assumption.

In general, the projections

$$p' = p \vee q - p \quad \text{and} \quad q' = p - p \wedge q$$

are orthogonal and equivalent by the Kaplansky parallelogram law [11, p. 403], via the projection $q - p \wedge q$. Consequently, by the preceding case we have $f(p') = f(q')$, which yields

$$f(p) = \frac{f(p \vee q) + f(p \wedge q)}{2}.$$

By symmetry we conclude that $f(p) = f(q)$. ■

There is a partial order on the set of all projections in the von Neumann algebra $\mathcal{M}$, which already entered into the preceding proof. We recall that $p \leqslant q$ if $pq = p$ and refer to [11, p. 110] for some other equivalent conditions. In this way the set of projections in $\mathcal{M}$ becomes a complete lattice [23, p. 69; 24, p. 290]. We say that a positive linear functional f on $\mathcal{M}$ is *subadditive* on the lattice of projections if

$$f(p \vee q) \leqslant f(p) + f(q)$$

for all projections p and q in $\mathcal{M}$. We first illustrate the use of the Lemma in the following characterization.

THEOREM 4. *A positive linear functional on a von Neumann algebra $\mathcal{M}$ is tracial if and only if it is subadditive on the lattice of projections in $\mathcal{M}$.*

Proof. If f is tracial, then its subadditivity is well known: due to the parallelogram law [11, p. 403], we have $f(p \vee q) - f(q) = f(p) - f(p \wedge q)$ and the subadditivity follows.

Conversely, let f be subadditive on the lattice of projections in $\mathcal{M}$. In view of the Lemma, we may assume that $\mathcal{M} = M_2(\mathbb{C})$. Let S be the self-adjoint matrix representing the positive functional f. That is, $f(A) = \operatorname{tr}(SA)$ for all A in $M_2(\mathbb{C})$. Suppose that the two eigenvalues λ and μ of S were different, and let u and v be the corresponding orthogonal eigenvectors. Let p_w denote the orthogonal projection onto the line $\mathbb{C}w$. Choose unit vectors x and y such that they are linearly independent (hence $p_x \vee p_y = I$) and close to v. More precisely, given $\varepsilon > 0$ we can do that with

$$|f(p_x - p_v)| < \varepsilon \quad \text{and} \quad |f(p_y - p_v)| < \varepsilon.$$

Since

$$\begin{aligned}\lambda + \mu = \operatorname{tr}(S) = f(I) &= f(p_x \vee p_y) \leqslant f(p_x) + f(p_y) \\ &\leqslant 2f(p_v) + 2\varepsilon = 2\mu + 2\varepsilon,\end{aligned}$$

we arrive at $\lambda \leqslant \mu + 2\varepsilon$. Since ε was arbitrary and the role of λ and μ is symmetric, we conclude that $\lambda = \mu$. Hence f is tracial. ∎

REMARK 1. It is well known that $f(p \vee q) \leqslant f(p) + f(q)$ holds for every positive functional f and every pair of commuting projections p and q; see [11, p. 168]. Theorem 4 shows that this inequality extends to all pairs of projections in a von Neumann algebra if and only if f is tracial. This in turn implies the following.

COROLLARY. *A von Neumann algebra is commutative if and only if every state is subadditive on the lattice of projections.*

Proof. By Theorem 4 every subadditive state is tracial; hence the commutativity follows by [11, Theorem 4.3.4(i)]. ∎

REMARK 2. The positivity of a bounded linear functional on a von Neumann algebra is a consequence of that property on the set of projections [23, p. 112]. Due to the subadditivity of the trace, some measure-theoretic arguments work in noncommutative probability. Theorem 4 shows that for nontracial functionals other methods are required.

THEOREM 5. *Let f be a linear functional on a C^*-algebra $\mathscr{A}$. The following are equivalent:*

(i) *f is positive and tracial;*

(ii) *for every positive integer k and every a in $\mathscr{A}$ we have $|f(a^k)| \leqslant f(|a|^k)$;*

(iii) *there exists a positive integer k such that $|f(a^k)| \leqslant f(|a|^k)$ for all a in $\mathscr{A}$.*

Proof. We first show that (i) implies (ii). Consider the canonical extension of f onto the second commutant of $\mathscr{A}$ in the GNS representation induced by f; see [24, p. 343] or [18, Lemma 3]. In this situation, (ii) becomes a special case of the Hölder inequality [4, Corollary 4.4(iii)]; see also [5] and [15]. We note that for even exponents an alternative elementary proof can be derived from some inequalities in [1, p. 90] and [2, p. 498]:

$$|f(a^{2k})| \leqslant f(|a^k|^2) \leqslant f(|a|^{2k}).$$

It remains to show that (iii) implies (i). Let $\mathscr{M}$ denote the second commutant of the GNS representation of $\mathscr{A}$ associated with the positive functional f; see [16, p. 47]. Then $\mathscr{M}$ is a von Neumann algebra, the strong closure of the image of $\mathscr{A}$; cf. [16, p. 22]. By the Kaplansky density theorem [16, p. 25] and the strong continuity of the functional calculus [24, p. 82], the inequality assumed in (iii) extends on $\mathscr{M}$. Finally, the Lemma and Theorem 1 imply the trace property. ∎

REFERENCES

1 H. Araki and S. Yamagami, An inequality for Hilbert-Schmidt norm, *Comm. Math. Phys.* 81:89–96 (1981).
2 J. C. Ault, An inequality for traces, *J. London Math. Soc.* 42:497–500 (1967).
3 F. F. Bonsall and J. Duncan, *Numerical Ranges of Operators on Normed Spaces and of Elements of Normed Algebras*, Cambridge U.P., Cambridge, 1971.
4 T. Fack, Sur la notion de valeur caractéristique, *J. Operator Theory* 7:307–333 (1982).
5 T. Fack and H. Kosaki, Generalized s-numbers of τ-measurable operators, *Pacific J. Math.* 123:269–300 (1986).
6 J. A. Feroe, A spectral radius theorem for matrix seminorms, *Linear Algebra Appl.* 22:97–103 (1978).
7 H. Flanders, Methods of proof in linear algebra, *Amer. Math. Monthly* 63:1–15 (1956).
8 L. T. Gardner, An inequality characterizes the trace, *Canad. J. Math.* 31:1322–1328 (1979).
9 I. C. Gohberg and M. G. Kreĭn, *Introduction to the Theory of Linear Nonselfadjoint Operators* (translation), Amer. Math. Soc., Providence, 1969.
10 P. D. Johnson, Spectral radius and seminorms in finite-dimensional algebras, *Colloq. Math.* 39:331–341 (1978).
11 R. V. Kadison and J. R. Ringrose, *Fundamentals of the Theory of Operator Algebras*, Vols. I, II, Academic, New York, 1983, 1986.
12 H. König, A formula for the eigenvalues of a compact operator, *Studia Math.* 65:141–146 (1979).

13 H. König, *Eigenvalue Distribution of Compact Operators*, Birkhäuser, Basel, 1986.
14 Z. V. Kovarik, Similarity and interpolation between projectors, *Acta Sci. Math. (Szeged)* 39:341–351 (1977).
15 V. I. Ovčinnikov, On the s-numbers of measurable operators (in Russian), *Funktsional. Anal. i Prilozhen.* 4(3):78–85 (1970).
16 G. K. Pedersen, *C*-Algebras and their Automorphism Groups*, Academic, London, 1979.
17 D. Petz, On spectral and central states of Banach algebras, *Acta Math. Hungar.* 42:19–24 (1983).
18 D. Petz, Spectral scale of self-adjoint operators and trace inequalities, *J. Math. Anal. Appl.* 109:74–82 (1985).
19 A. Pietsch, *Eigenvalues and s-Numbers*, Geest & Portig, Leipzig, 1987.
20 M. Reed and B. Simon, *Methods of Modern Mathematical Physics IV: Analysis of Operators,* Academic, New York, 1978.
21 J. R. Retherford, Trace, in *Special Topics of Applied Mathematics* (Proc. Sem., Ges. Math. Datenverarb., Bonn, 1979), North-Holland, Amsterdam, 1980, pp. 47–56.
22 K. Shoda, Einige Satze über Matrizen, *Japan. J. Math.* 13:361–365 (1936).
23 Ş. Strătilă and L. Zsidó, *Lectures on von Neumann Algebras*, Editura Academiei, Bucharest, 1979.
24 M. Takesaki, *Theory of Operator Algebras I*, Springer, New York, 1979.
25 H. K. Wimmer, Spectral radius and radius of convergence, *Amer. Math. Monthly* 81:625–627 (1974).
26 J. Zemánek, Idempotents in Banach algebras, *Bull. London Math. Soc.* 11:177–183 (1979).
27 G. K. Pedersen and E. Størmer, Traces on Jordan algebras, *Canad. J. Math.* 34:370–373 (1982).

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